Canonical rings of stacky surfaces

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January 2024

The one ring

The canonical ring of a stack $\mathscr{X}$ is

$$R(\mathscr{X}) := \bigoplus_{d=0}^{\infty} H^0(\mathscr{X}, \omega_{\mathscr{X}}^{\otimes d})$$

where $\omega_{\mathscr{X}} = \bigwedge^{\dim \mathscr{X}} \Omega_{\mathscr{X}}$. [1]

Reasons for interest -

- X a curve, $g \geq 2 \implies \operatorname{Proj} R(X) \simeq X$. (Petri, 1923)
- X a variety $\implies \kappa(X) = \dim \operatorname{Proj} R(X)$.
- $R(X)$ f.g. $\implies \operatorname{Proj} R(X)$ is a minimal model for X. (Birkar, Cascini, Hacon, McKernan 2010)
- Models of $\mathscr{X}$ in weighted projective space.

[1] By Peter J. Yost - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=98351026

Rings of power(s)

More generally, if $\mathscr{F}$ is a sheaf on $\mathscr{X}$, can consider

$$R(\mathscr{X}, \mathscr{F}) := \bigoplus_{d=0}^{\infty} H^0(\mathscr{X}, \mathscr{F}^{\otimes d})$$

Important special case - Δ is NCD. and $\mathscr{F} = \omega_{\mathscr{X}}(\log \Delta)$.

- $\mathscr{X}$ a modular curve, Δ the cusp divisor $\implies$ $R(\mathscr{X}, \omega_{\mathscr{X}}(\log \Delta))$ graded ring of modular forms. (Voight, Zureick-Brown 2022)
- $\mathscr{X}$ a smooth compactification of a moduli space with NCD boundary, Δ the boundary $\implies$ $R(\mathscr{X}, \omega_{\mathscr{X}}(\log \Delta))$ graded ring of automorphic forms.
  - $\mathscr{X} = \mathscr{A}_g$ (PPAV) for Siegel modular forms.
  - $\mathscr{X}$ classifies RMAV for Hilbert modular forms.

Birational invariant

When X is a variety, $R(X)$ is a birational invariant.

We say that $f : \mathscr{X} \to \mathscr{Y}$ is strongly birational if the induced morphism $X \to Y$ is birational and

$$\begin{array}{ccc} \mathscr{X} & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is a fiber square.

Proposition (A., Chidambram, Frengley, Schiavone, Webb)

If $f : \mathscr{X} \to \mathscr{Y}$ is a strongly birational morphism of tame DM stacks with smooth projective coarse moduli, then the induced map of canonical rings is an isomorphism.

Stacky surfaces

A stacky surface $\mathscr{X}$ over a field k is a smooth proper geom. connected DM stack of dimension 2 over k with an open subscheme $i : U \hookrightarrow \mathscr{X}$ such that $\dim(\mathscr{X} \setminus U) = 0$.

A log stacky surface $(\mathscr{X}, \Delta)$ is a stacky surface $\mathscr{X}$ equipped with a NC divisor $\Delta \subseteq U$.

The signature of $(\mathscr{X}, \Delta)$ is

$$(h^{\bullet,\bullet}(X); (e_1; a_1, b_1), \ldots, (e_r; a_r, b_r); h^{\bullet}(\Delta)),$$

where $(e_i; a_i, b_i)$ are the orders and rotation types of the r stacky points, so that locally $(z_1, z_2) \mapsto (\zeta_{e_i}^{a_i} z_1, \zeta_{e_i}^{b_i} z_2)$.

Goal

Conjecture

Let $(\mathscr{X}, \Delta)$ be a tame log stacky surface over a perfect field k such that $K_{\mathscr{X}} + \Delta$ is ample. Let $e = \max(1, e_1, \ldots, e_r)$. Then $R(\mathscr{X}, \Delta)$ is generated in degree at most $5e$ with relations in degree at most $10e$.

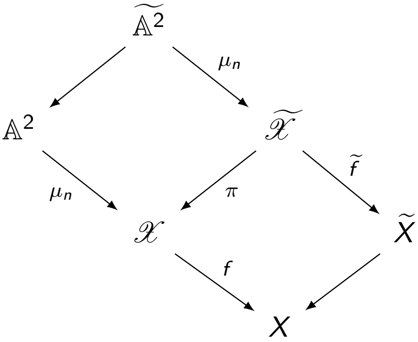
Conjecture

If $(K_{\mathscr{X}} + \Delta)^2 \geq 3$, $h^0(K_{\mathscr{X}} + \Delta) \geq 2$, $h^1(K_{\mathscr{X}} + \Delta) = 0$, and $\mathscr{X}$ has an irreducible stacky curve $\mathscr{C} \in |K_{\mathscr{X}} + \Delta|$ then $R(\mathscr{X}, \Delta)$ is generated in degree at most $3e$ with relations in degree at most $6e$.

Other than some special cases, expect degrees to depend only on the signature σ.

Start with X a variety, and proceed inductively. (Voight, Zureick-Brown 2022)

Handling codim 2 stacky points



- $X = \mathbb{A}^2/\mu_n = \text{Spec}(k[x,y]^{\mu_n})$
- $\widetilde{X}$ is a toric resolution of singularities
- $\mathscr{X} = [\mathbb{A}^2/\mu_n]$ has stacky structure only above the origin
- $\widetilde{\mathscr{X}}$ is root stack over the resolution cycle in $\widetilde{X}$

Relation to $\mathbb{Q}$-divisors

Assume P is a stacky point of type $(n; a, b)$, let

$$L = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \lambda_1 a + \lambda_2 b \equiv 0 \bmod n\}$$

and let $\frac{1}{n}(l_i, m_i)$ be the boundary points of the positive cone in $L^\vee$.

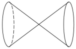
Theorem (A., Chidambaram, Frengley, Schiavone, Webb)

There exist exceptional divisors E_i in $\widetilde{\mathscr{X}}$, pointwise fixed by μ_n such that

$$\pi^* K_{\mathscr{X}} = f^* K_{\widetilde{X}} + \sum_i (n - l_i - m_i) E_i.$$

Example - diagonal action $(n; 1, 1)$

- $\mu_n \curvearrowright \mathbb{A}^2$ via $(x, y) \mapsto (\zeta_n x, \zeta_n y)$.
- $X = \operatorname{Spec} k[x, y]^{\mu_n} = \operatorname{Spec} k[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n]$.



- $\widetilde{\mathbb{A}^2} = \operatorname{Bl}_0 \mathbb{A}^2$, with $(x, y), (s : t) \mapsto (\zeta_n x, \zeta_n y), (s : t)$.
- $\widetilde{X}$ is the cylinder based on E on which μ_n acts trivially.

RH for f + contraction $\pi \implies$

$$\pi^* K_{\mathscr{X}} + E = K_{\widetilde{\mathscr{X}}} = f^* K_{\widetilde{X}} + (n-1) \cdot E,$$

hence $\pi^* K_{\mathscr{X}} = f^* K_{\widetilde{X}} + (n-2) \cdot E$, so that

$$h^0(\mathscr{X}, dK_{\mathscr{X}}) = h^0\left(\widetilde{X}, dK_{\widetilde{X}} + \left\lfloor d\left(1 - \frac{2}{n}\right)\right\rfloor E\right).$$

Previous work

- Conjectures hold for X an algebraic surface ($e = 1$) with $\Delta = 0$. (Ciliberto 1983, Reid 1988)
- Section rings of $\mathbb{Q}$-divisors on minimal rational surfaces (Landesman, Ruhm, Zhang 2018) are related via birational maps, using the previous theorem.
- Spin canonical rings of log stacky curves (Landesman, Ruhm, Zhang 2016) are related via the hyperplane section principle.

Spin canonical rings

Proposition (The hyperplane section principle - Reid, 1988)

Let R be a graded ring, $x_0 \in R_{a_0}$ regular $a_0 > 0$, $\overline{R} = R/(x_0)$. If $\overline{R} = k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ then

$$R = k[x_0, x_1, \dots, x_n]/(F_1, \dots, F_n),$$

where F_i is such that $\deg F_i = \deg f_i$ and $F(0, x_1, \dots, x_n) = f(x_1, \dots, x_n)$.

Assume there is an irreducible stacky curve $\mathscr{C} \in |K_{\mathscr{X}} + \Delta|$. Let $A = (K_{\mathscr{X}} + \Delta)|_{\mathscr{C}}$. By adjunction

$$K_{\mathscr{C}} + \Delta|_{\mathscr{C}} = (K_{\mathscr{X}} + O_{\mathscr{X}}(\mathscr{C}) + \Delta)|_{\mathscr{C}} = 2(K_{\mathscr{X}} + \Delta)|_{\mathscr{C}} = 2A,$$

so that A is a log spin-canonical divisor on $\mathscr{C}$.

Base case - log canonical rings

Theorem (A., Chidambaram, Frengley, Schiavone, Webb)

If X is regular, $K_X + \Delta$ is ample and $p_a(\Delta) = 1$, then $R(X, \Delta)$ is generated in degree at most 5 with relations in degree at most 10.

Proof. (Sketch)

The closed subscheme exact sequence for Δ yields

$$0 \to O_X(-\Delta) \to O_X \to O_\Delta \to 0$$

twisting by $d(K_X + \Delta)$ leads to

$$0 \to (d-1)(K_X+\Delta)+K_X \to d(K_X+\Delta) \to (K_X+\Delta)|_\Delta = K_\Delta \to 0.$$

from LES, using $K_X + \Delta$ ample (and X regular for $d = 1$) get

$$h^0(d(K_X + \Delta)) = \chi((d-1)(K_X + \Delta) + K_X) + h^0(dK_\Delta)$$

Base case - log canonical rings

Proof. (Sketch).

Riemann-Roch and adjunction give

$$\chi((d-1)(K_X+\Delta)+K_X)$$
$$=\chi+\frac{1}{2}d(d-1)(K_X+\Delta)^2-(d-1)(p_a(\Delta)-1)-\delta_{d=1}$$

When $p_a(\Delta)=1$, then $K_\Delta=0$, so if $h=h^0(\Delta)$, get

$$h^0(d(K_X+\Delta))=\chi+h-\delta_{d=1}+\frac{1}{2}d(d-1)(K_X+\Delta)^2.$$

Writing $g=\chi+h-1$ and $c=(K_X+\Delta)^2$, get Hilbert series

$$\Phi(R;t)=(1+(g-3)t+(c-2(g-2))t^2+(g-3)t^3+t^4)(1-t)^{-3}$$

+ the base-point-free pencil trick. □

Induction step

Birational map $\mathscr{X} \to \mathscr{X}'$ where

$$Q \in \mathscr{X} \mapsto P' \in \mathscr{X}'$$

with degree $e \geq 2$.

$R = R(\mathscr{X}, \Delta)$ is an R'-algebra for $R' = R(\mathscr{X}', \Delta)$

Goal

Explicit description of generators and relations for R' over R.

Proposition

Assume Q is of type $(e; 1, 1)$. For $3 \leq i \leq e$, a general choice of $y_i \in H^0(\mathscr{X}, i(K_{\mathscr{X}} + \Delta))$ minimally generates R as an R'-algebra.

Tools - Riemann-Roch, adjunction and ampleness of $K_{\mathscr{X}} + \Delta$.

Issues - $K_{\mathscr{X}'} + \Delta$ might not be ample.

Application - Hilbert modular surfaces

Let F be a real quadratic field. Consider the moduli space $\mathscr{Y}_F$ of abelian surfaces A with RM by $\mathbb{Z}_F$.

Theorem (Rapoport, 1978)

$\mathscr{Y}_F$ admits a compactification $\mathscr{X}_F$ with boundary Δ such that $(\mathscr{X}, \Delta)$ is a log stacky surface. $R(\mathscr{X}, \Delta)$ is the graded ring of Hilbert modular forms of parallel even weight.

Moreover, we have $p_a(\Delta) = 1$ and

Theorem (Baily-Borel, 1966)

In the above setting, $K_{\mathscr{X}} + \Delta$ is ample.

Base case - Hilbert modular surfaces

Example

Let $F = \mathbb{Q}(\sqrt{5})$, and consider the moduli space $Y_{F,2}$ of abelian surfaces A with RM by $\mathbb{Z}_F$ and a basis for $A[2]$. Then $X_{F,2}$ is a (non-stacky) surface.
One computes that $h^0(\Delta) = 5$, $(K_X + \Delta)^2 = 8$ and $\chi = 1$, so

$$\Phi(R; t) = \frac{1 + 2t + 2t^2 + 2t^3 + t^4}{(1-t)^3} = \frac{1 - t^2 - t^4 + t^6}{(1-t)^5},$$

corresponding to

$$R(X, \Delta) = k[x_1, \ldots, x_5]/(f_2, f_4).$$

This matches an explicit description (van der Geer, 1980).